

CS103, AUTUMN 2026



Lecture 09:

Graph Theory (Part 1 of 3)

CS198 Applications!

*See additional slide deck on
today's lecture page for
information about applying to
SL for CS106A and CS106B!*

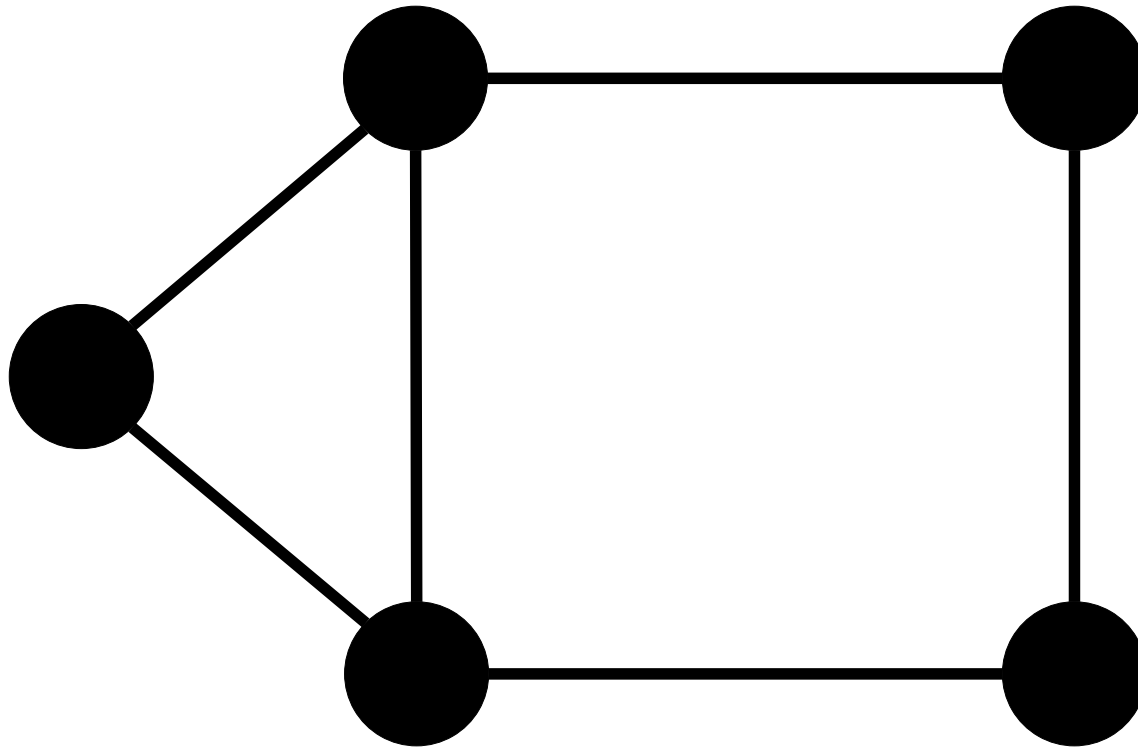
Graphs

Part 1

- 1. Graph Overview**
2. Examples
3. Graphs and Digraphs
4. Formalizing Graphs
5. Announcements
6. Vertex Covers
7. Independent Sets
8. A Proof on Graphs
9. Recap and What's Next?

Graph Overview

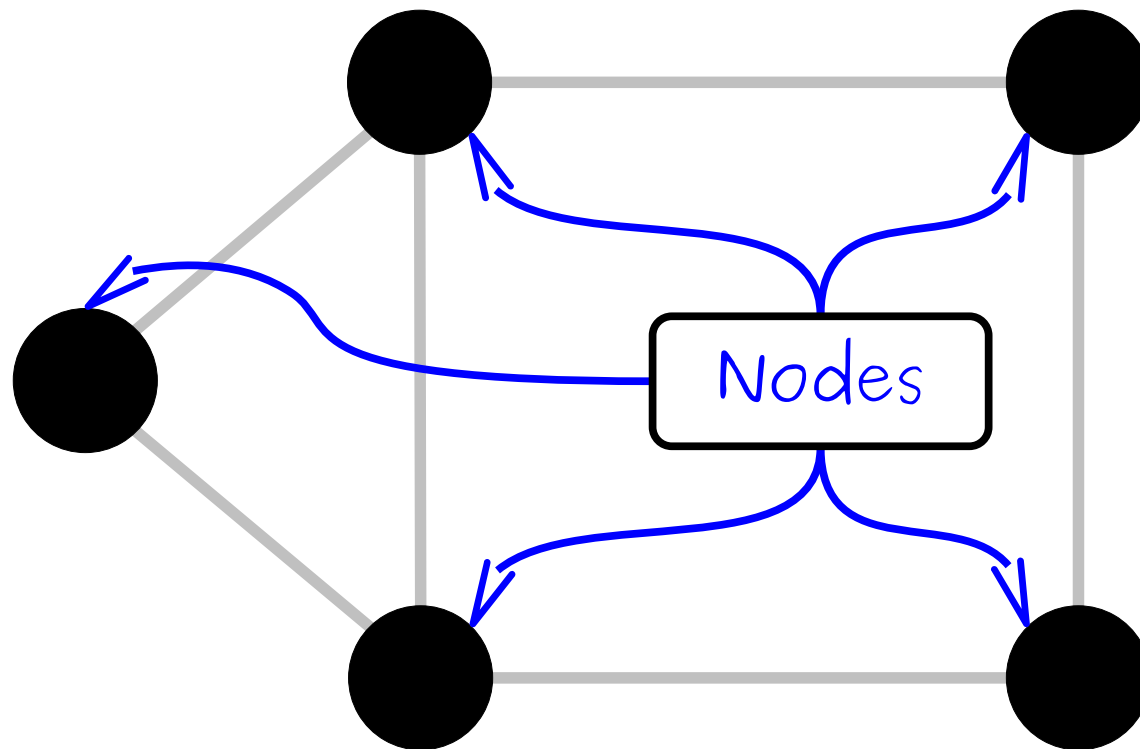
A **graph** is a mathematical structure for representing relationships.



A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**)

Graph Overview

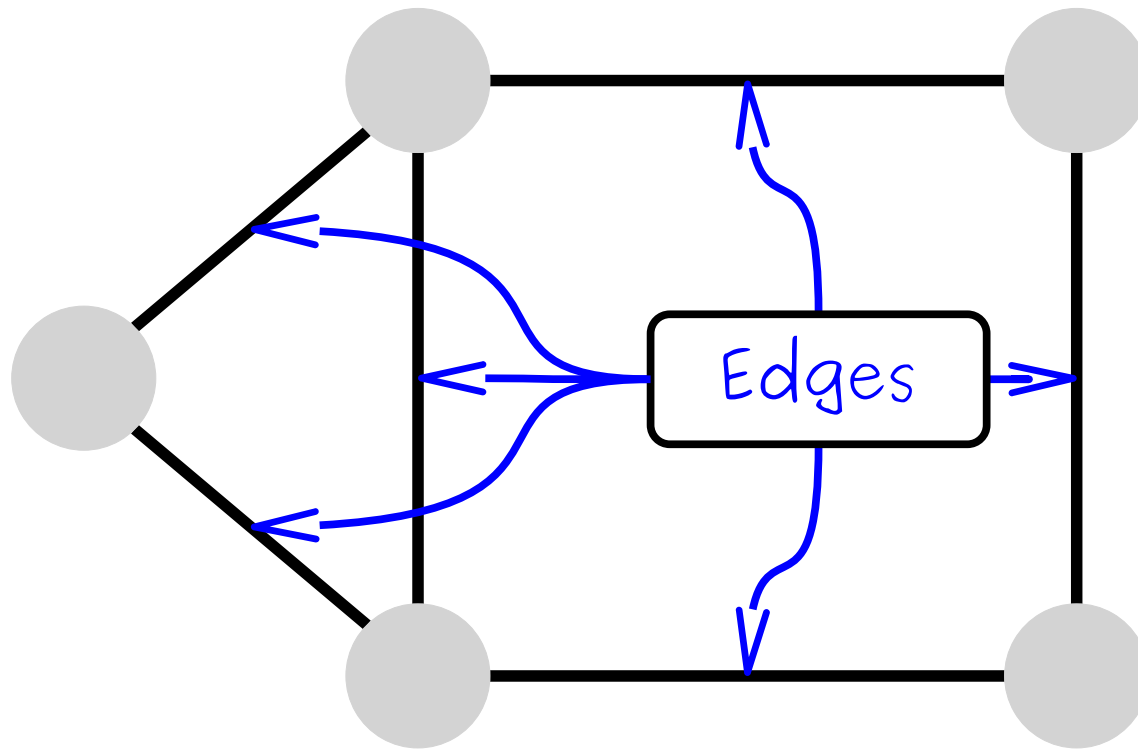
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Graphs

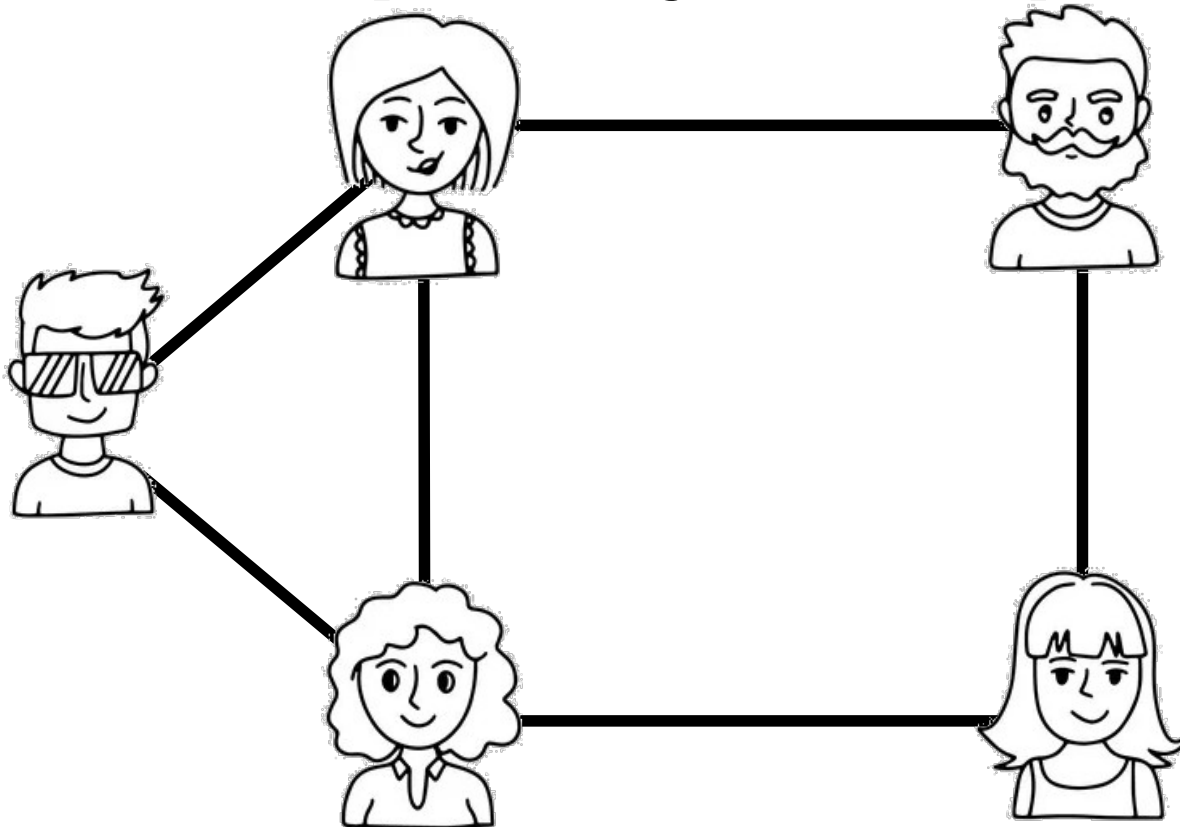
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facebook

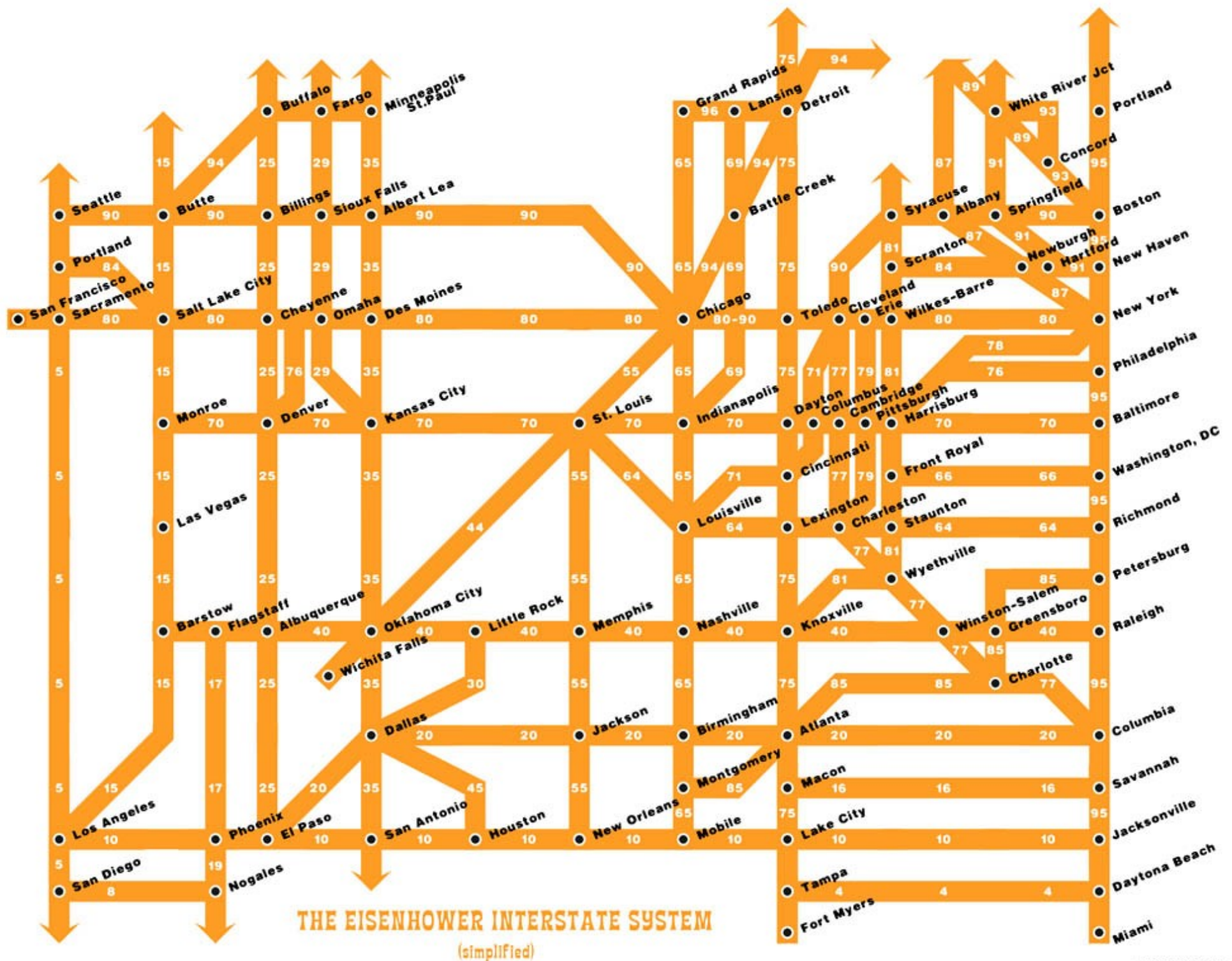
LinkedIn

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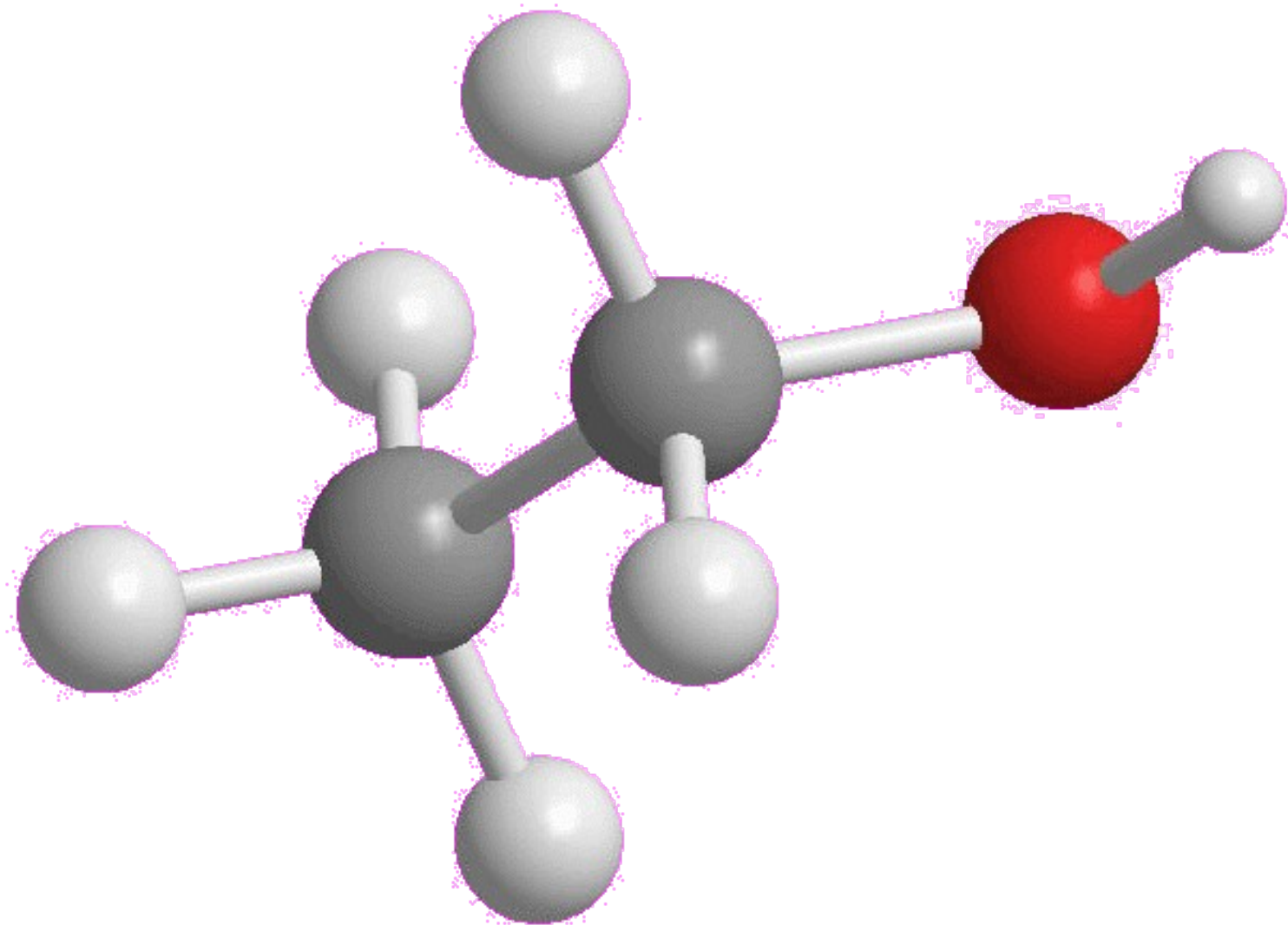


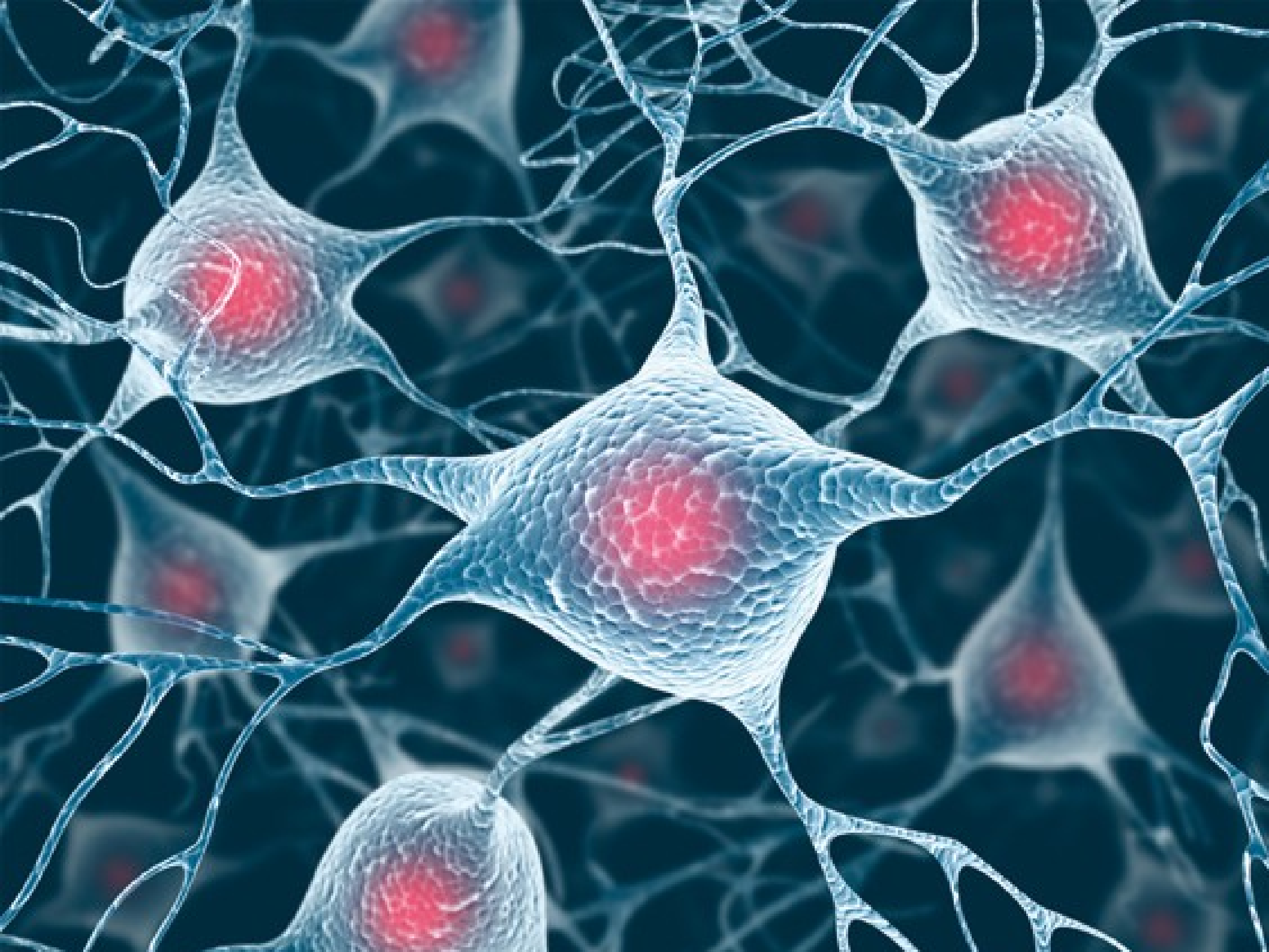
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Chemical Bonds





Commonalities

- Each of these structures consists of
 - a collection of objects and
 - links between those objects.
- ***Today's Goal:*** Develop a general framework for describing structures like these that generalizes the idea across a wide domain.

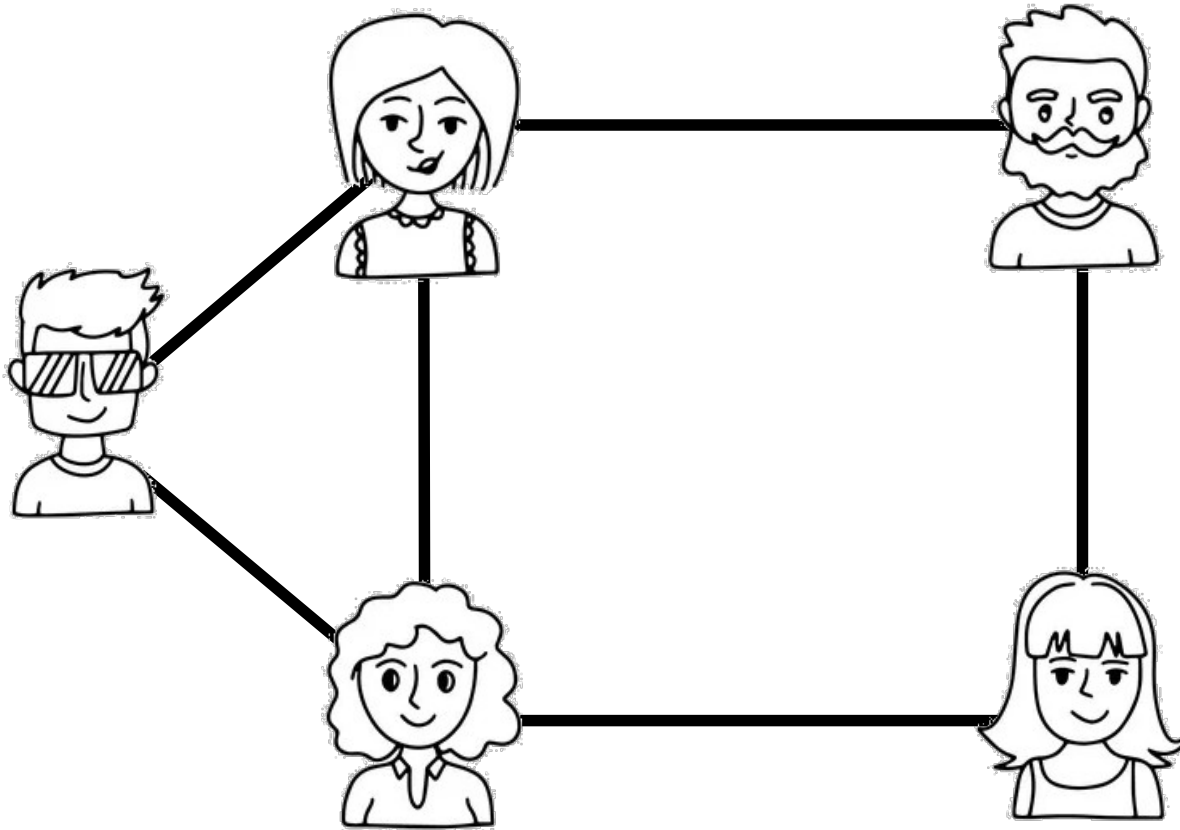
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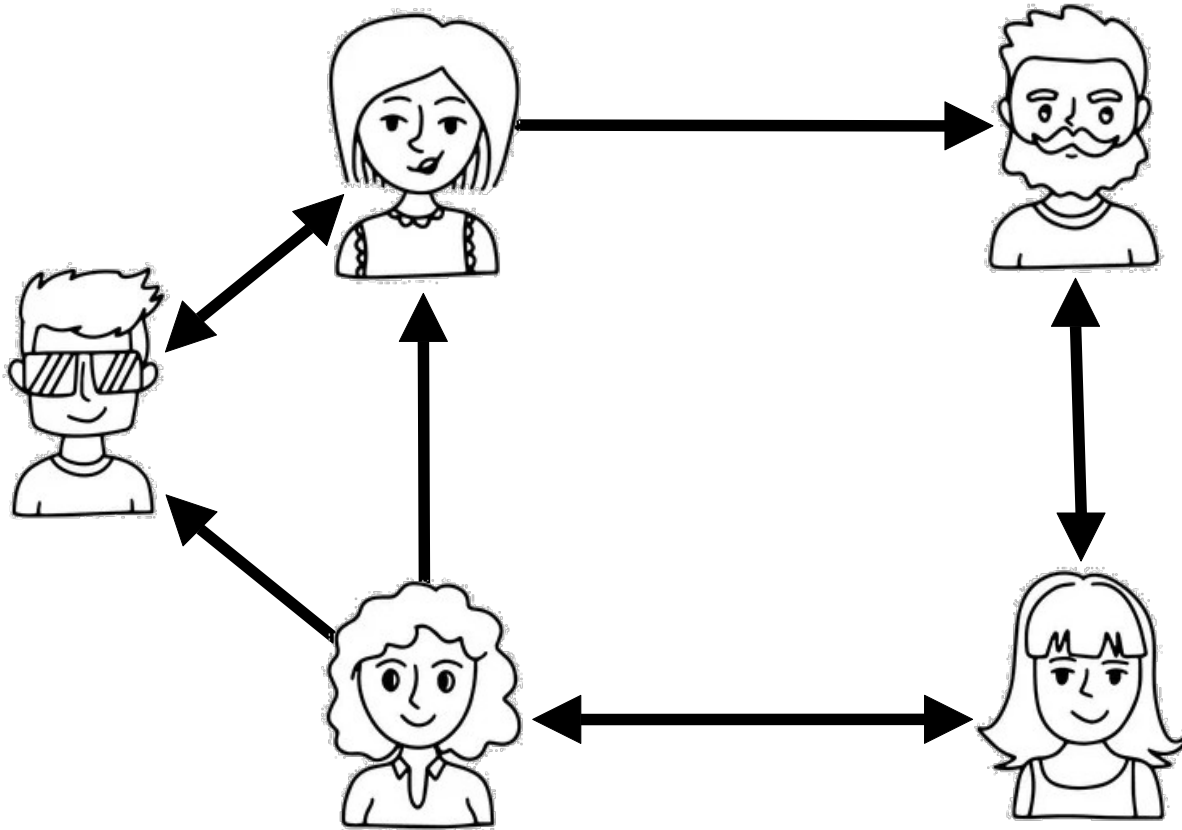
Graphs and Digraphs

Some graphs are *undirected*.



Graphs and Digraphs

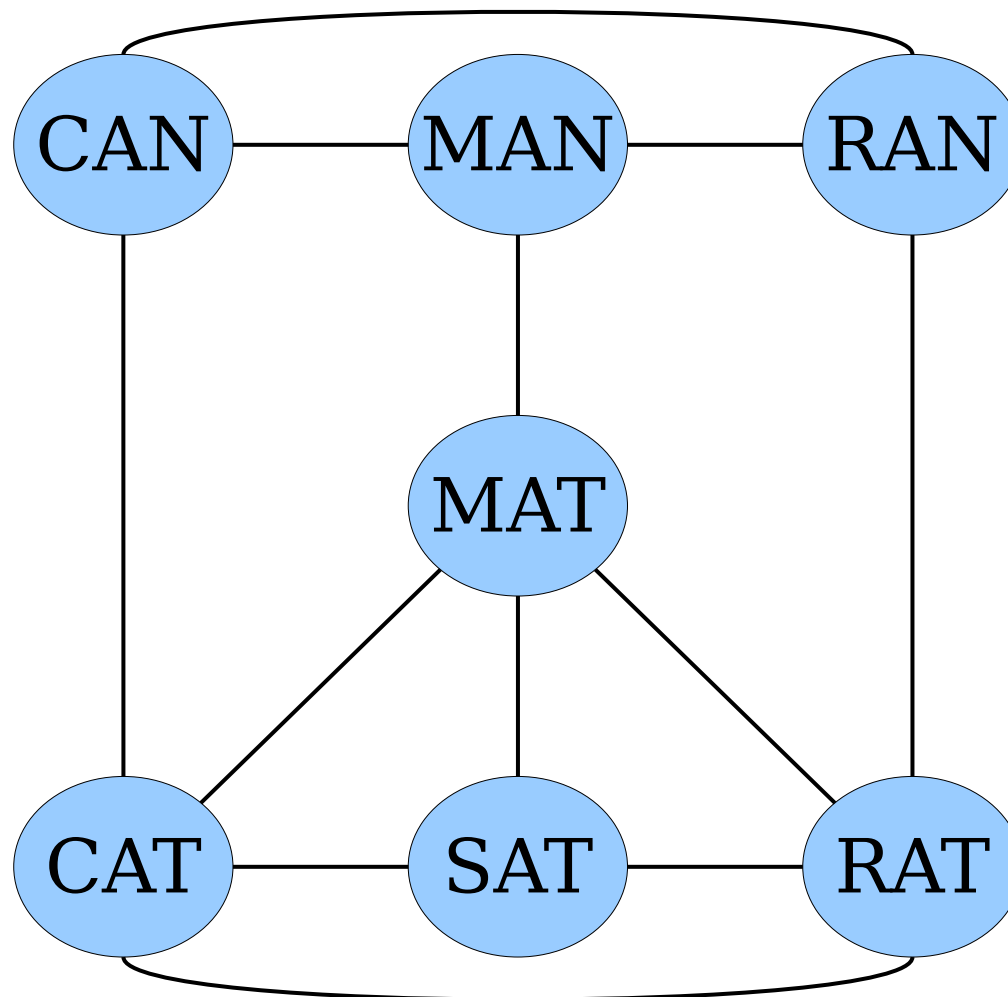
Some graphs are *undirected*.



Some graphs are *directed*.

Graphs and Digraphs

Should this graph be *directed* or *undirected*?



Graphs and Digraphs

- An **undirected graph** is one where edges link nodes, with no endpoint preferred over the other.
- A **directed graph** (or **digraph**) is one where edges have an associated direction.
 - (There's something called a **mixed graph** that allows for both types of edges, but they're fairly uncommon and we won't talk about them.)
- Unless specified otherwise:

👉 ***“Graph” means “undirected graph”*** 👈

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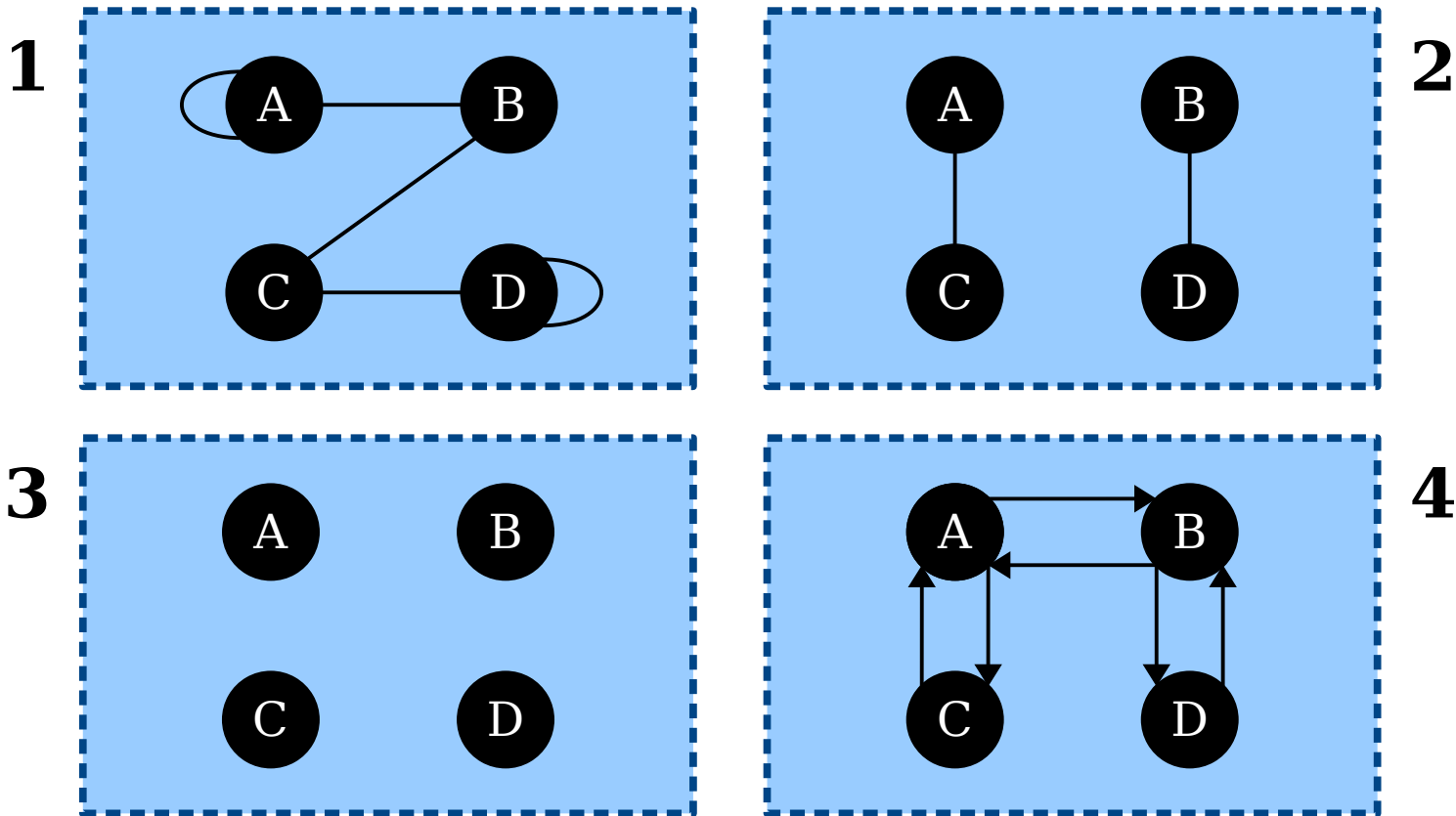
Formalizing Graphs

- How might we define a graph mathematically?
- We need to specify
 - what the nodes in the graph are, and
 - which edges are in the graph.
- The nodes can be pretty much anything.
- This is pretty broad, but that's a good thing!
- What about edges?

Formalizing Graphs

- An **unordered pair** is a set $\{a, b\}$ of two elements $a \neq b$. (Remember that sets are unordered.)
 - For example, $\{0, 1\} = \{1, 0\}$
- An **undirected graph** is an ordered pair $G = (V, E)$, where
 - V is a set of nodes, which can be anything, and
 - E is a set of edges, which are *unordered* pairs of nodes drawn from V .
- A **directed graph** (or **digraph**) is an ordered pair $G = (V, E)$, where
 - V is a set of nodes, which can be anything, and
 - E is a set of edges, which are *ordered* pairs of nodes drawn from V .

- An **unordered pair** is a set $\{a, b\}$ of two elements $a \neq b$.
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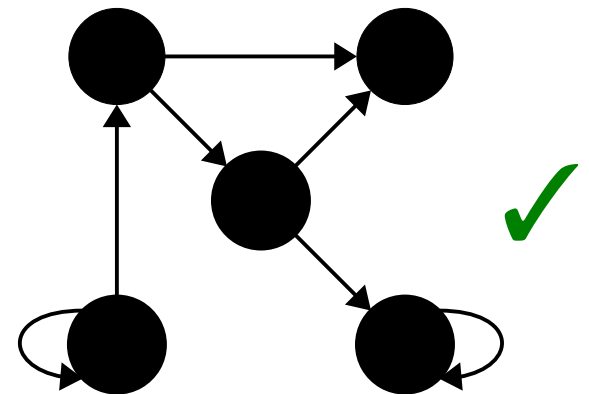
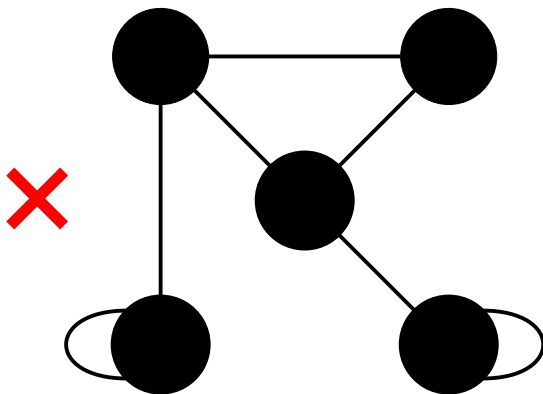
Which of these are drawings of undirected graphs?

Answer at

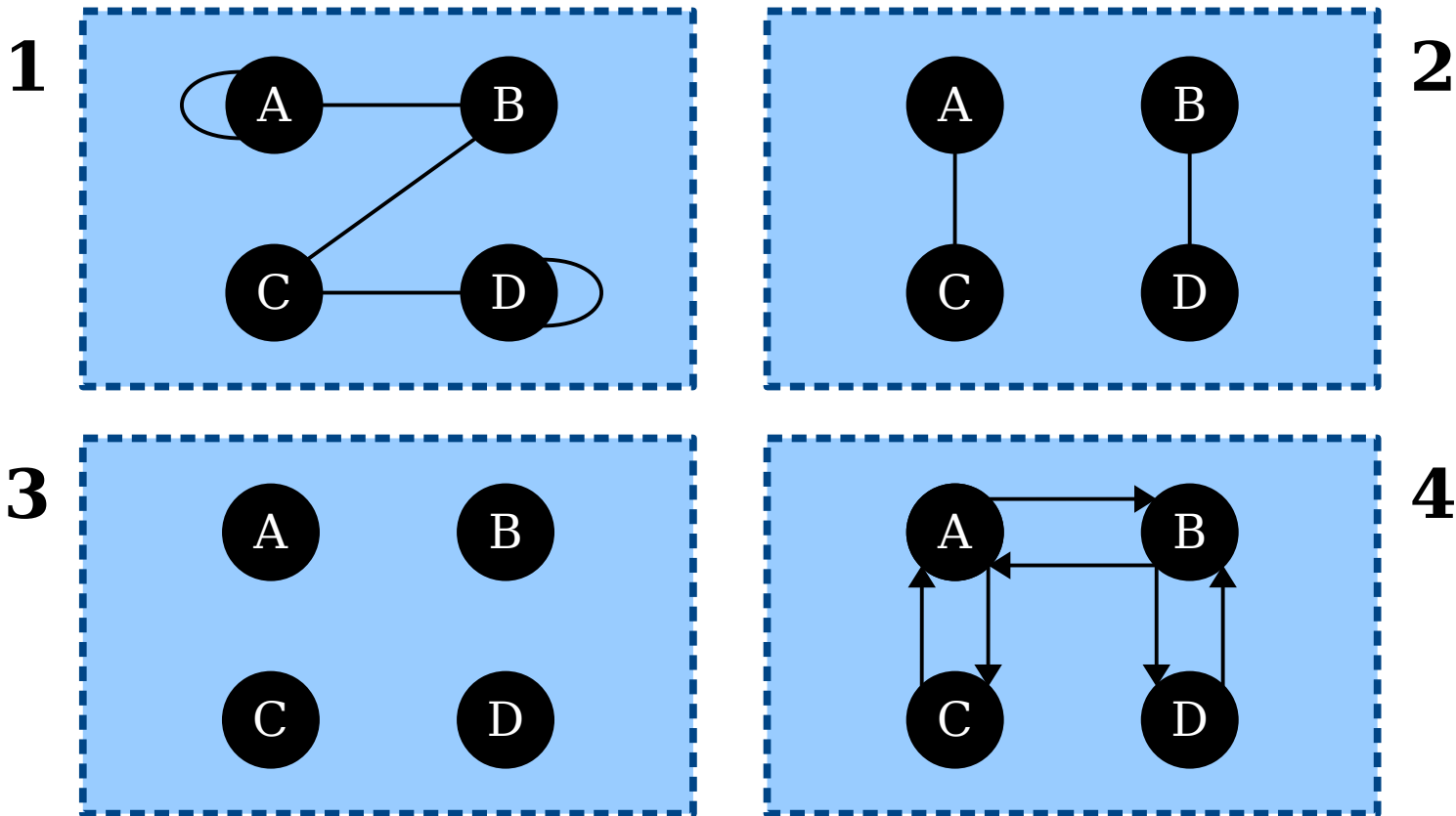
<https://cs103.stanford.edu/pollev>

Self-Loops

- An edge from a node to itself is called a ***self-loop***.
- In (undirected) graphs, self-loops are generally not allowed.
 - Can you see how this follows from the definition?
- In digraphs, self-loops are generally allowed unless specified otherwise.



- An **unordered pair** is a set $\{a, b\}$ of two elements $a \neq b$.
- An **undirected graph** is an ordered pair $G = (V, E)$, where
 - V is a set of nodes, which can be anything, and
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Which of these are drawings of undirected graphs?

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PS2 Solutions Released

- We don't release solutions to autograded problems.
- If you have any questions about those, ping us privately over EdStem or come talk to us at our office hours.

 CS103 

Problem Set 2

**Due Friday, October 9 at 1:00 PM
Pacific**

Solutions are available!

 [Solutions](#)

This second problem set explores mathematical logic and dives deeper into formal mathematical proofs. We've chosen the questions here to help you get a more nuanced understanding for what first-order logic statements mean (and, importantly, what they don't mean) and to give you a chance to practice your proofwriting. By

PS3 Due Friday

- PS3 is due this Friday at 1:00 PM.
 - **Ask questions** if you have them! That's what we're here for. You can ask on EdStem or in office hours.
 - If you haven't yet started, **please do so today**. That will give you time to digest the concepts and read up on anything you're still confused about.
 - Please **tag problems and partners**. :)

Midterm Exam Logistics

- Our first midterm exam is next **Monday, October 19th**, from **7-10 PM**. Locations vary.
- You're responsible for Lectures 00 – 08 and topics covered in PS1 – PS3. This includes the assume-prove table. As far as functions go, if you're solid on the first-order logic definitions and the application of the assume-prove table to function proofs, you should be fairly set for how functions will appear on the exam. Exam problems may build on the written or coding components from the problem sets.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.

Midterm Exam Logistics

- Students with alternate exam arrangements: you should have heard from us with details. If you haven't, contact Joyce and Sean ASAP.
- We have posted an ***Exam Logistics*** page on the course website with full details and logistics. We added a few details and practice problems this past weekend.
- The page also includes advice from former CS103 students about how to do well on the exams.
- Check it out. There are tons of goodies there to help you prepare for the exam.

Exam Day Logistics

- We'll have proctors in the room.
- We have assigned seating; see course website. These will be posted sometime tomorrow (Tuesday, Oct. 13).
- Check out your seat in advance, and screenshot it!
- No phones, calculators, or other digital devices during the exam.
- No belongings near your seats (leave them in your dorms or at the front of the classroom).
- Must have Stanford student ID to turn in exam.
- We will at some point take a picture of the exam room.

Review Session

- Two of our amazing CAs, Benjy and Mia, will hold a review session this **Friday, October 16th**, time TBD. More details will be posted on Ed soon!
 - The review session will be recorded.
- Come prepared to discuss any questions you may have.
- You'll get more out of this session if you have done some preliminary study first.

Participation Opt-Out

- Participation opt-out due Friday, 11:59 PM.
- This is a firm deadline.

Left-Handed Desk Form

- This is posted on Ed and is due tonight (Monday) by 11:59 PM.

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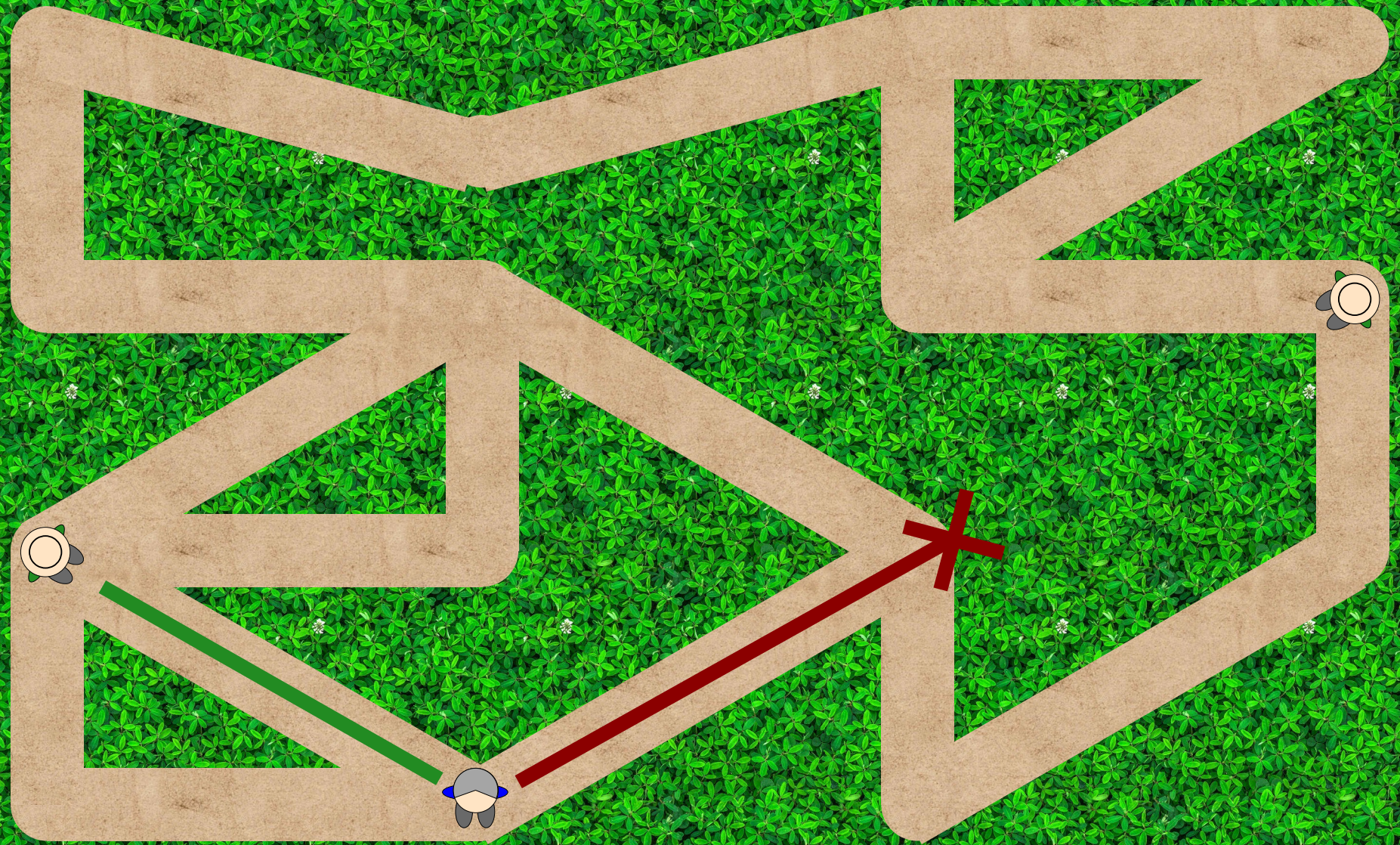
Vertex Covers

- Let $G = (V, E)$ be an undirected graph. A **vertex cover** of G is a set $C \subseteq V$ such that the following statement is true:

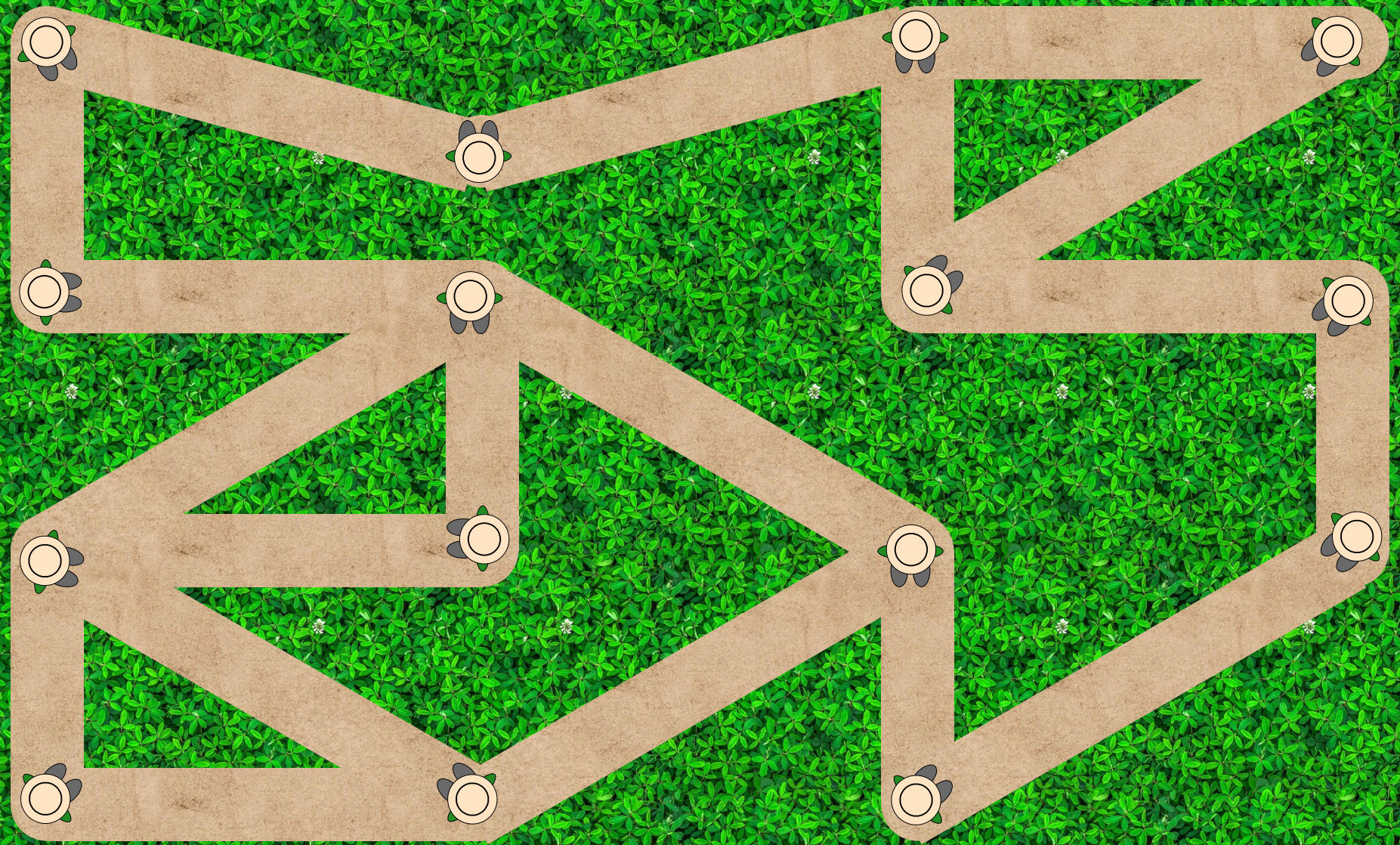
$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow (u \in C \vee v \in C))$$

(“Every edge has at least one endpoint in C .”)

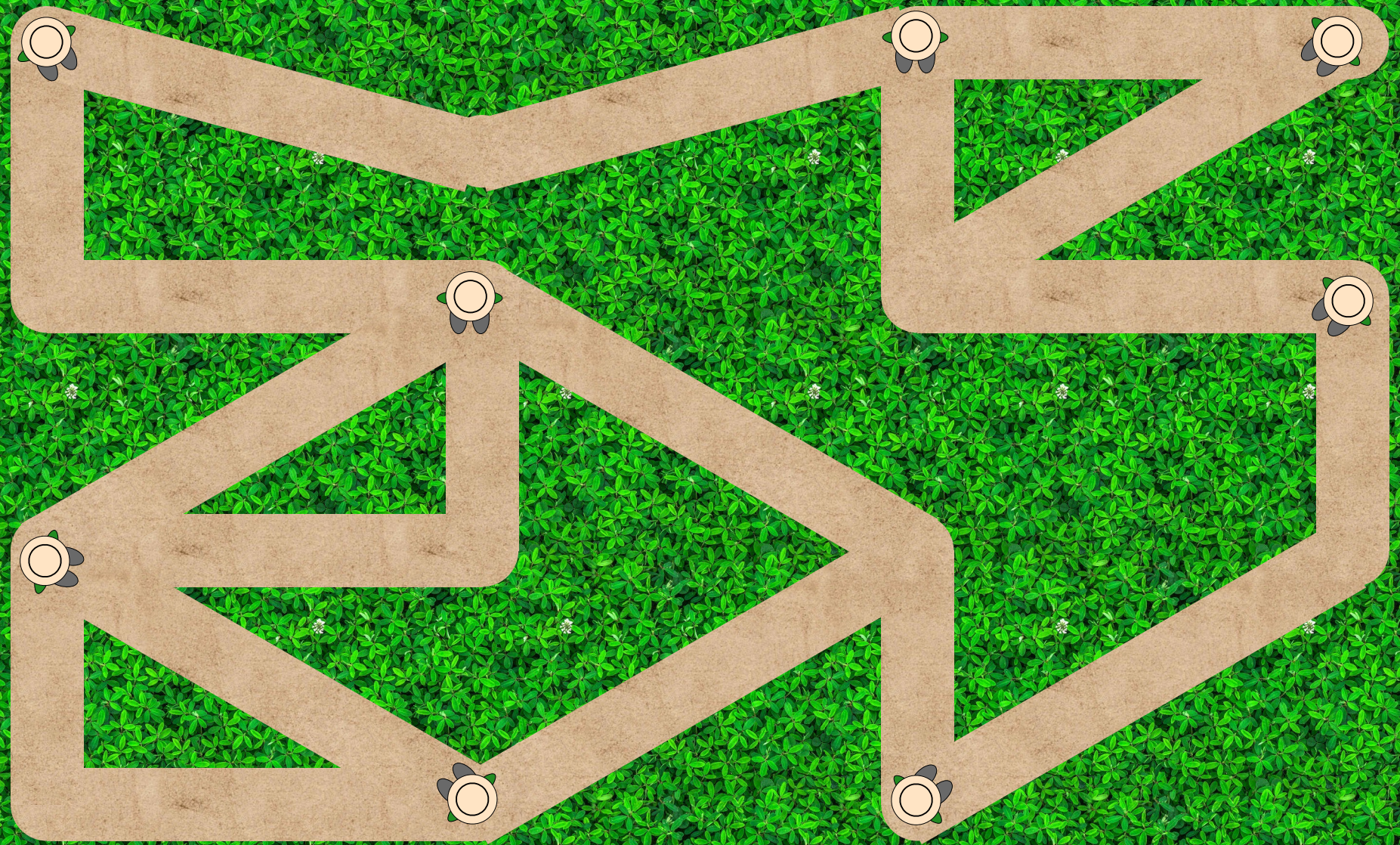
- Intuitively speaking, a vertex cover is a set formed by picking at least one endpoint of each edge in the graph.
- Vertex covers have applications to placing streetlights/benches/security guards, as well as in gene sequencing, machine learning, and combinatorics.



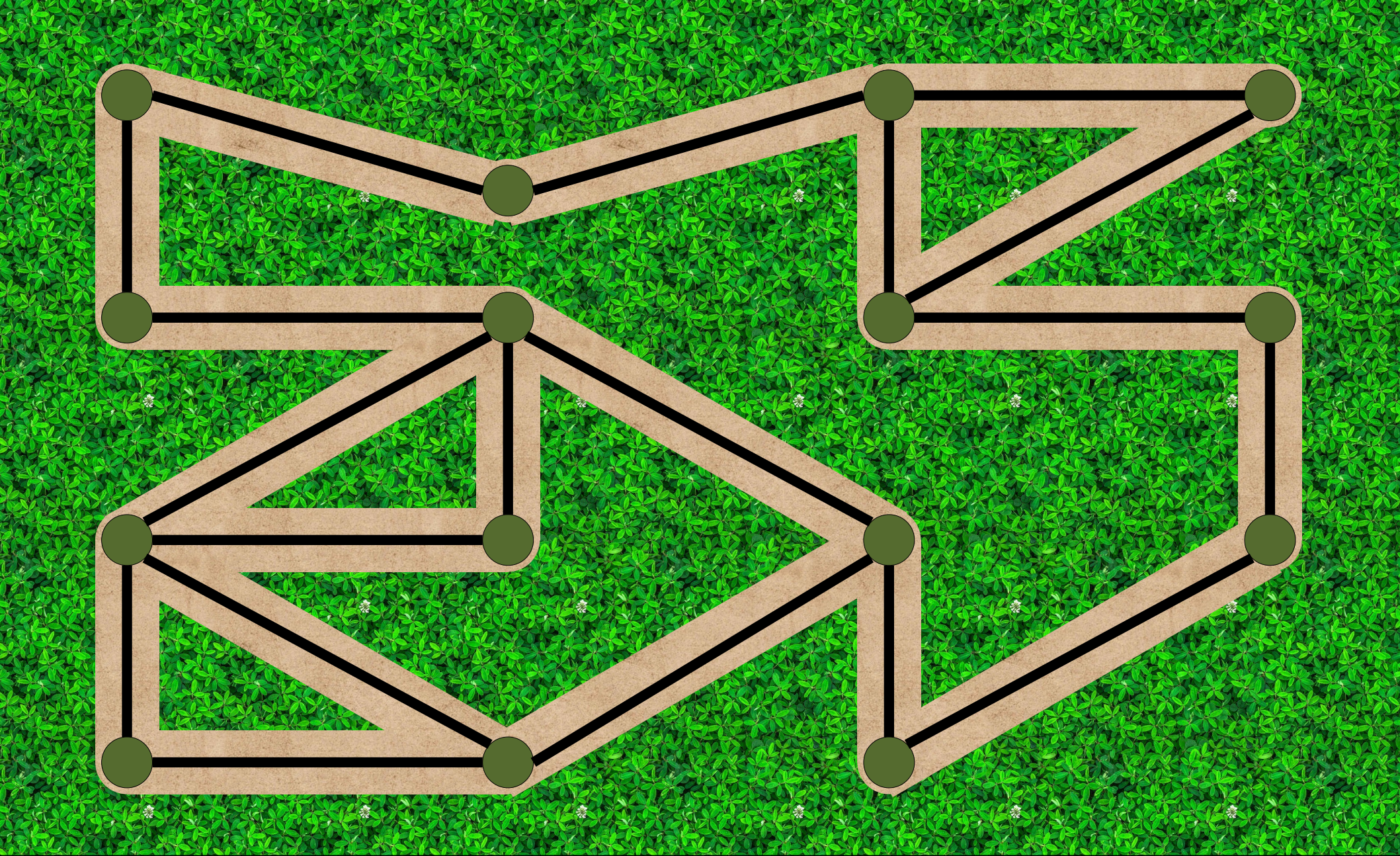
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.



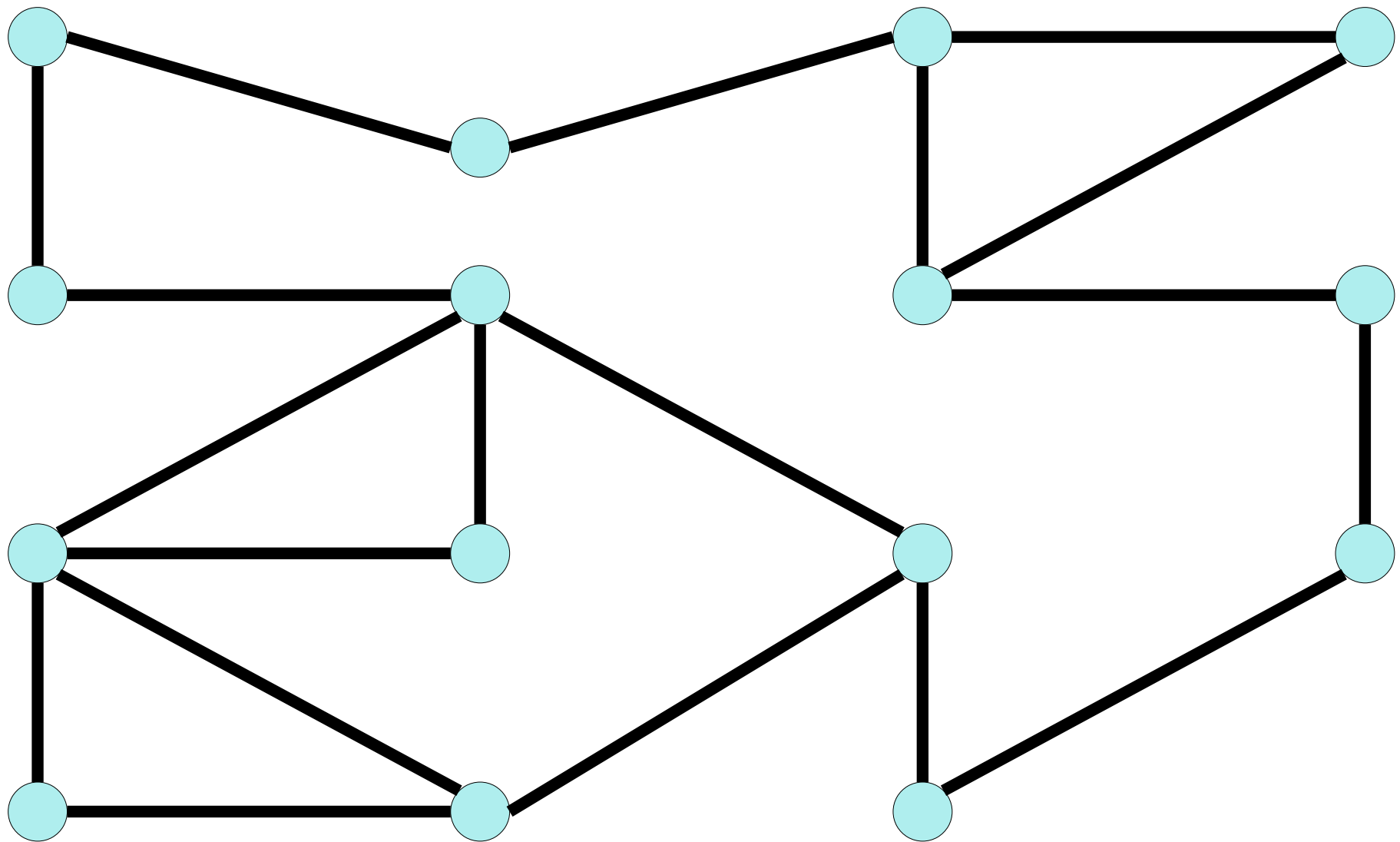
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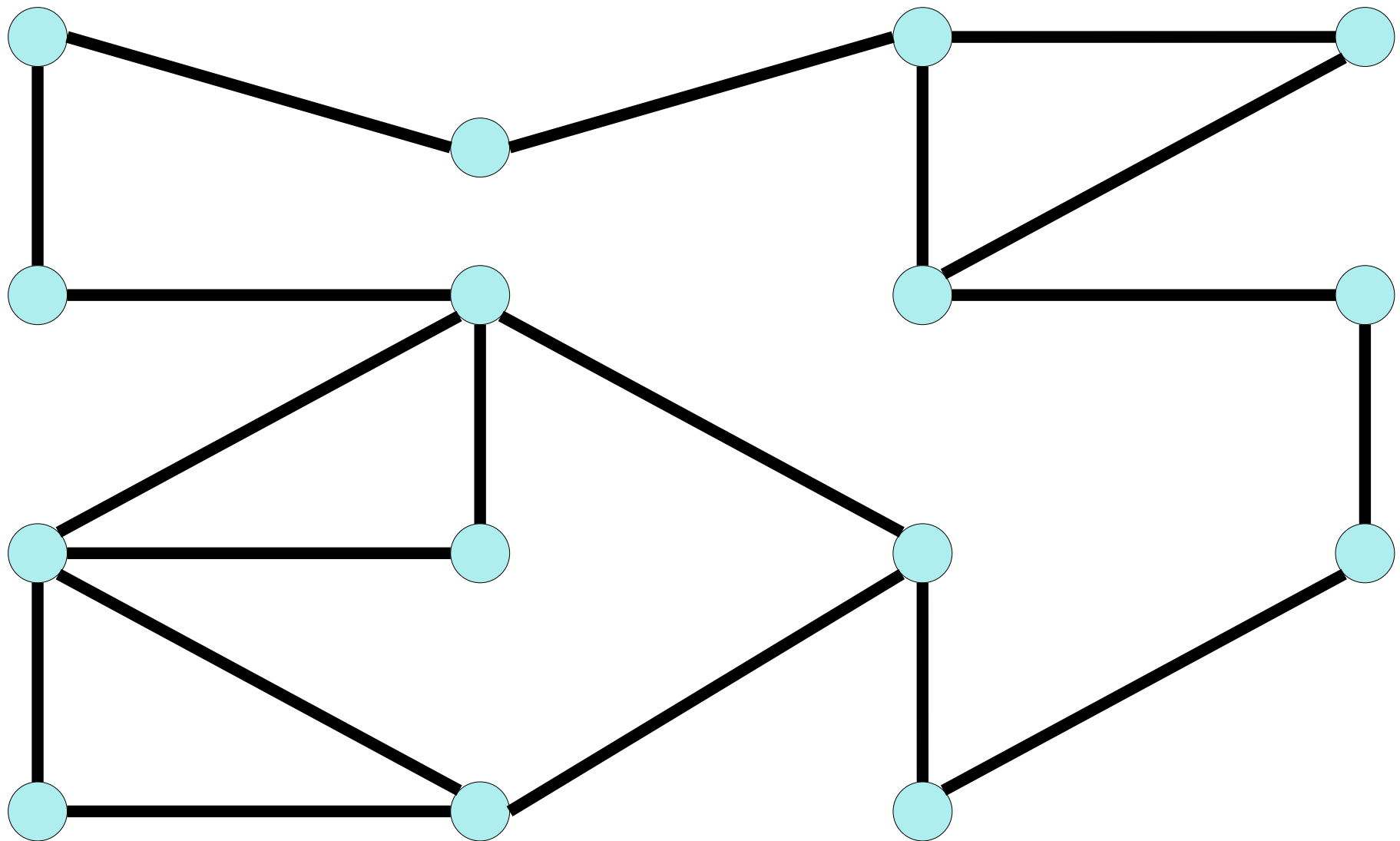
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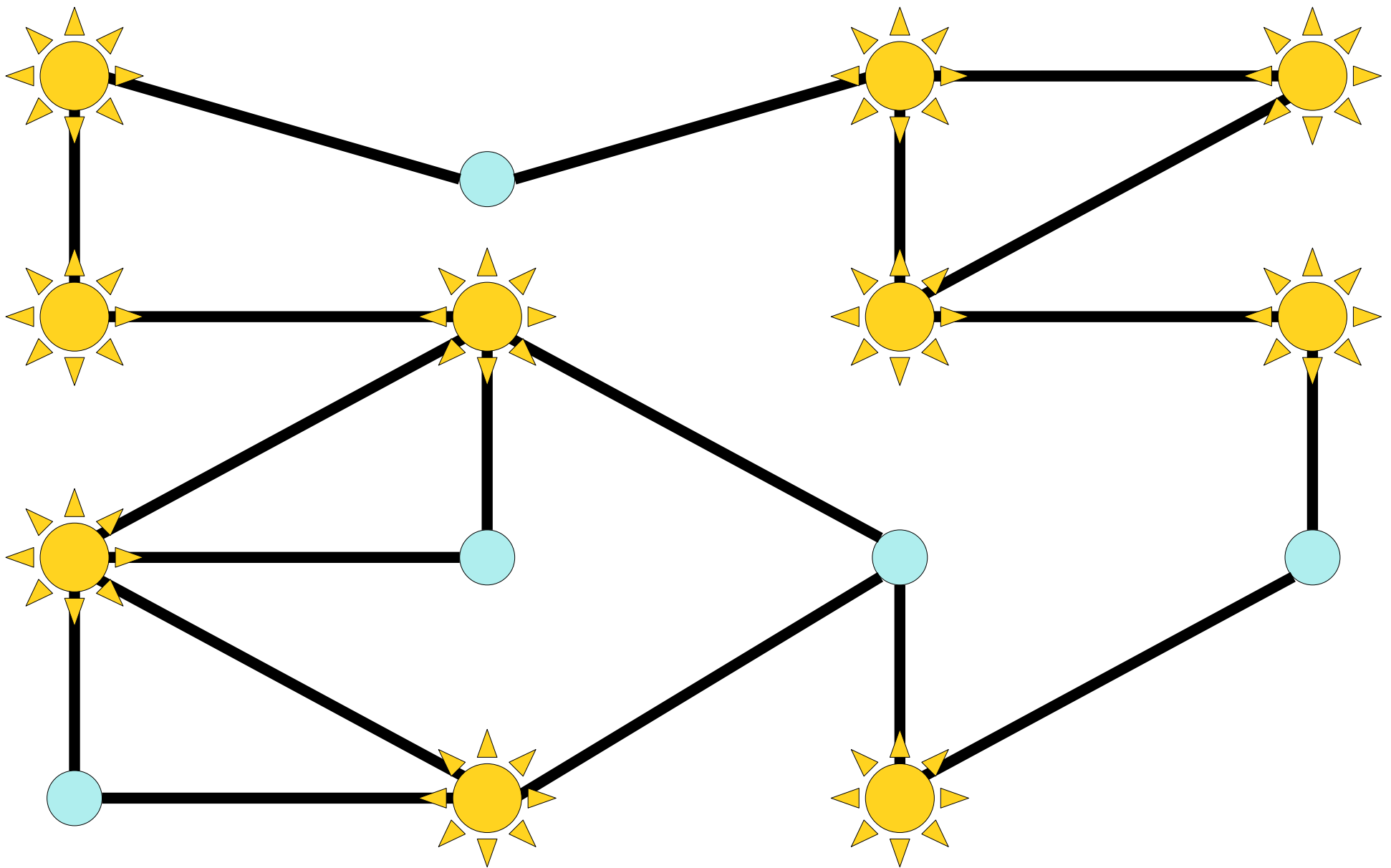
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Choose at least one endpoint of each edge.



Choose at least one endpoint of each edge.

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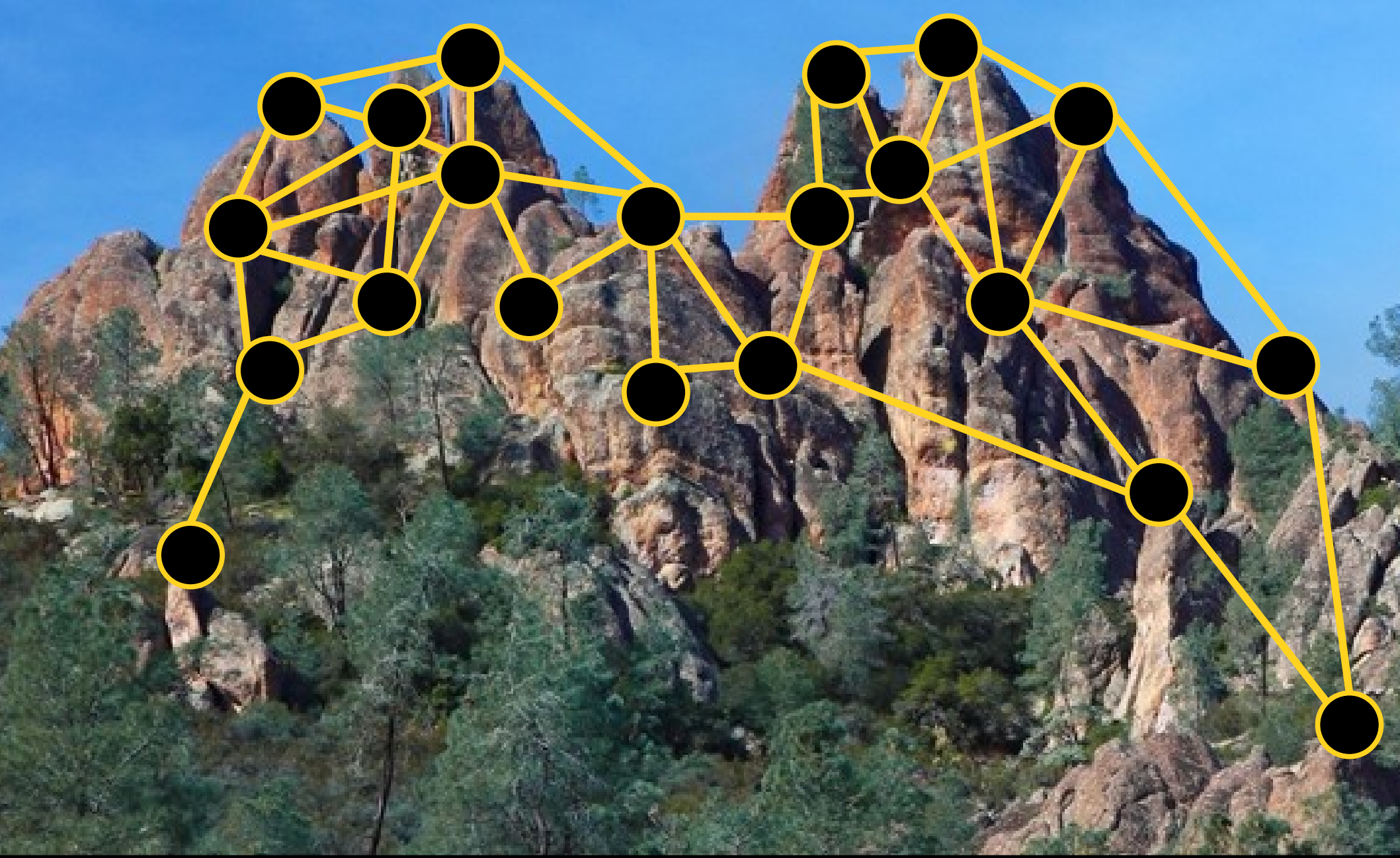
Independent Sets

- If $G = (V, E)$ is an (undirected) graph, then an **independent set** in G is a set $I \subseteq V$ such that

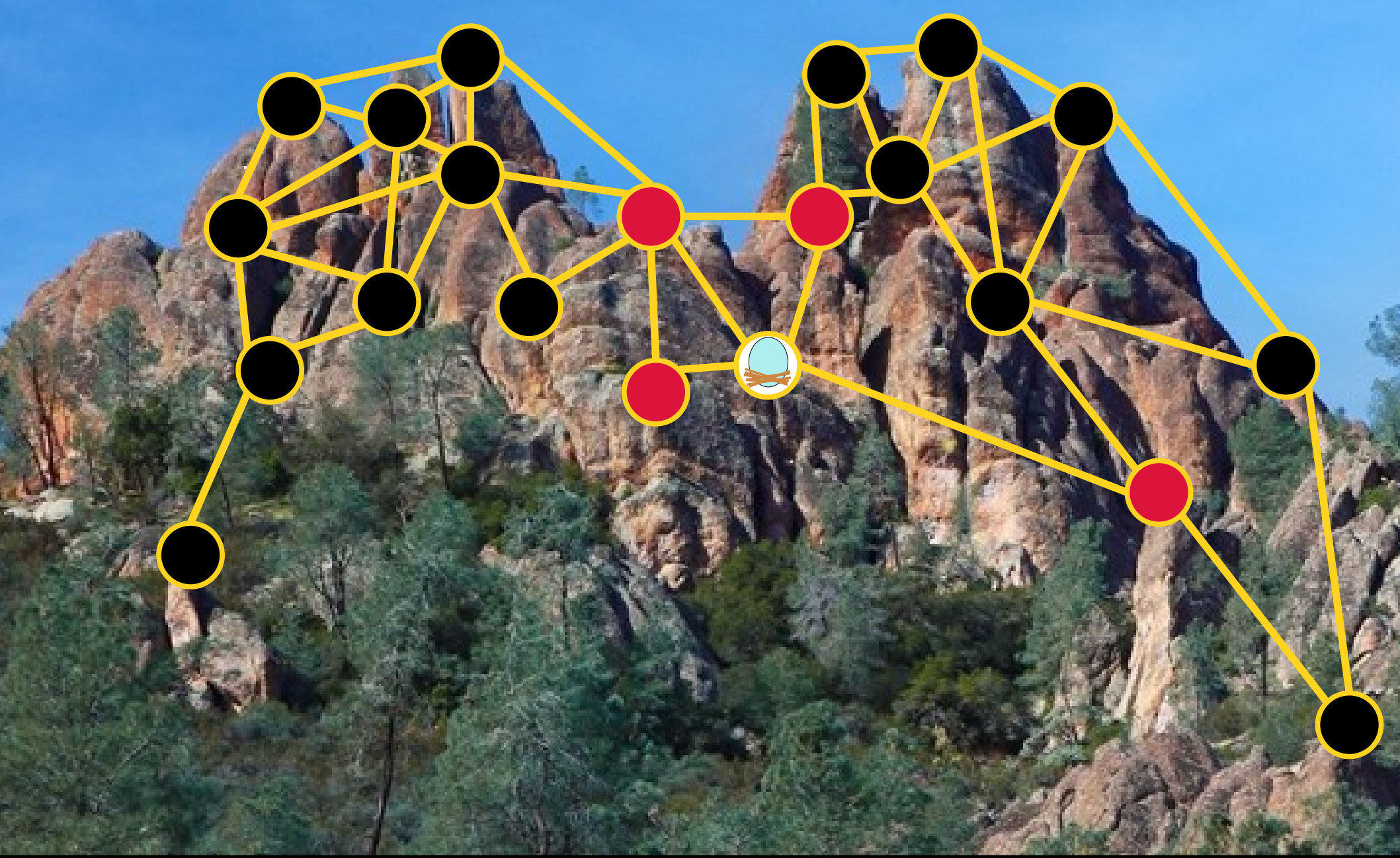
$$\forall x \in I. \forall y \in I. \{x, y\} \notin E.$$

(“No two nodes in I are adjacent.”)

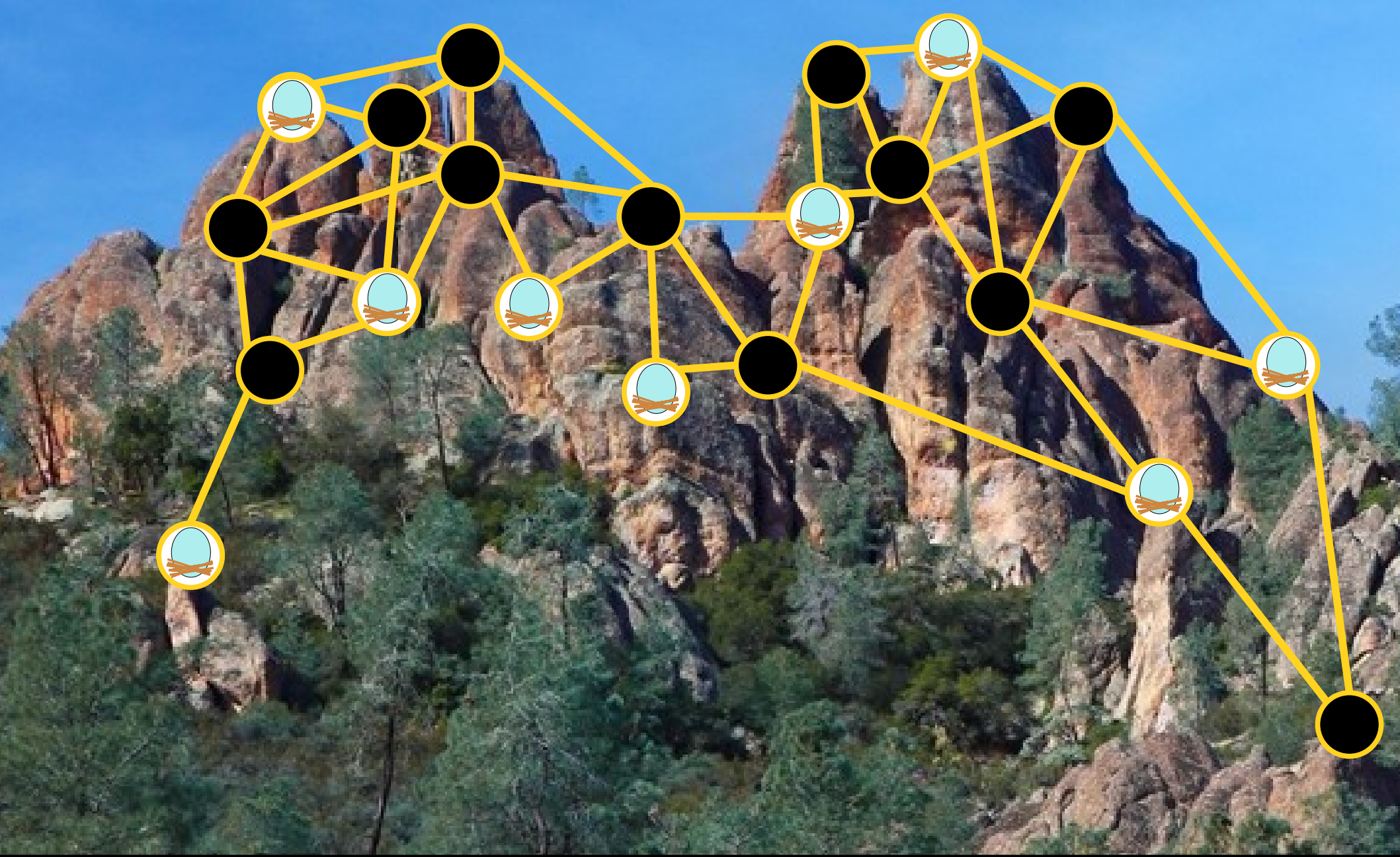
- Independent sets have applications to resource optimization, conflict minimization, error-correcting codes, cryptography, and more.



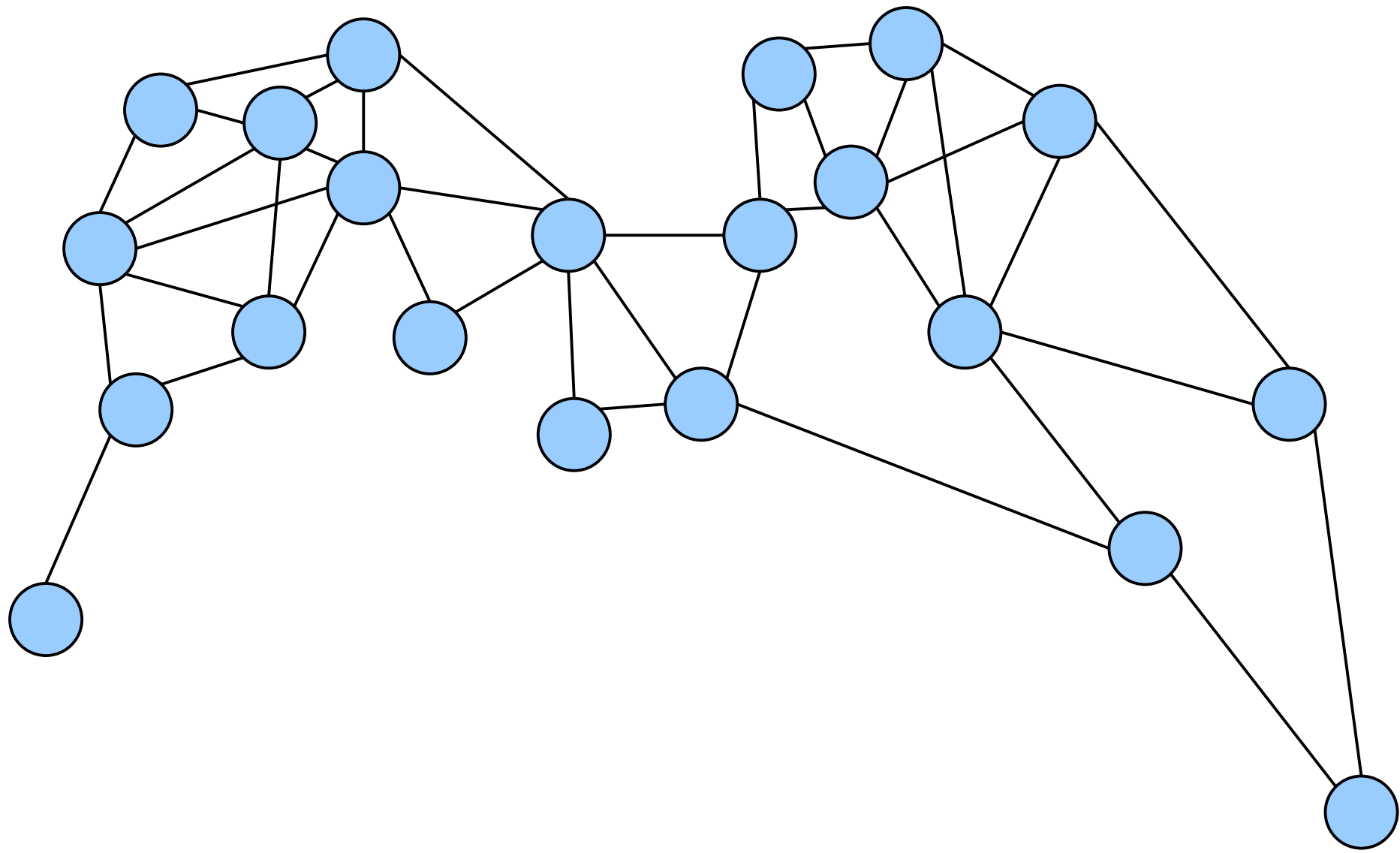
Set up nests for the California condor. Condors are territorial and won't nest if they can see other condors.



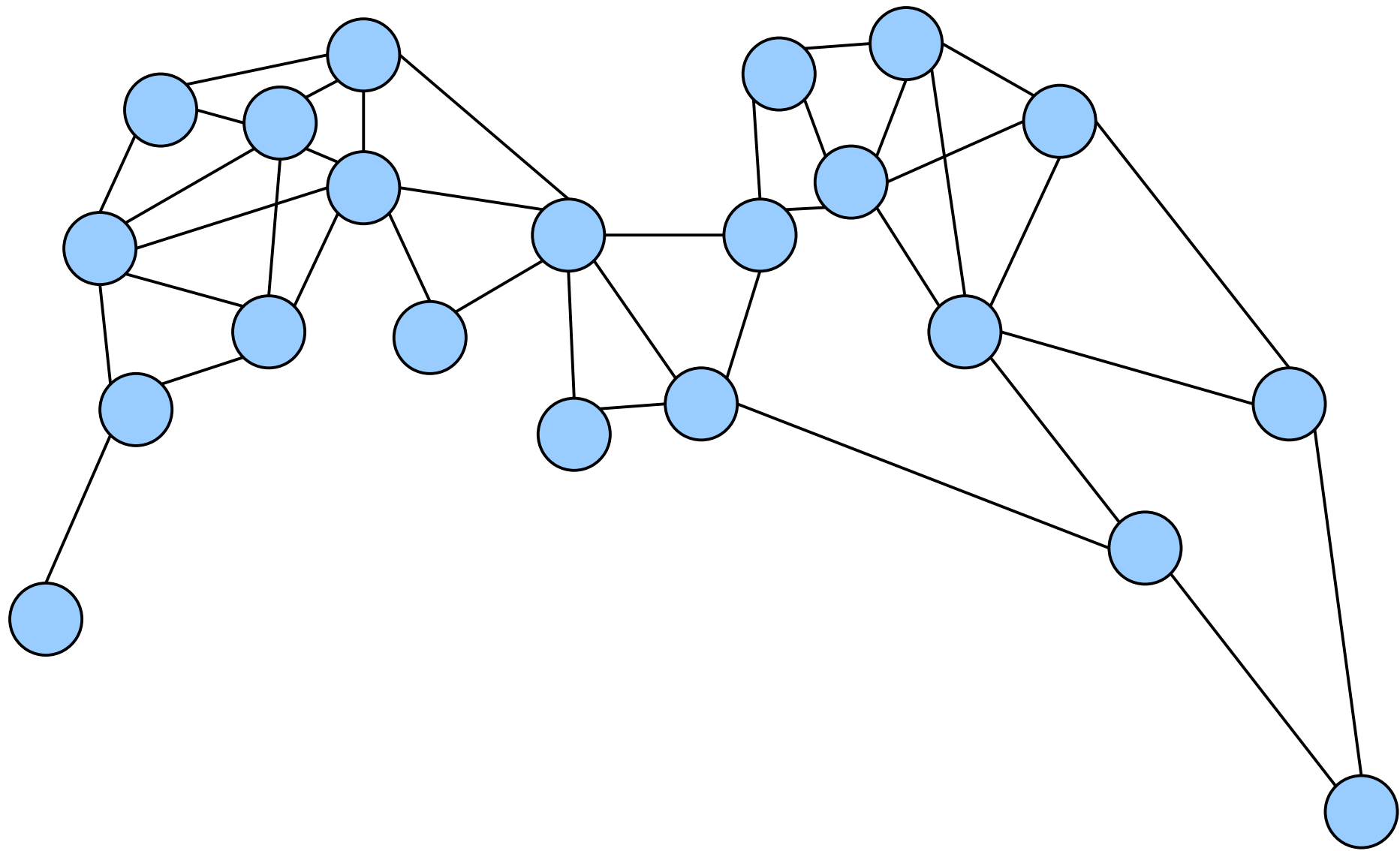
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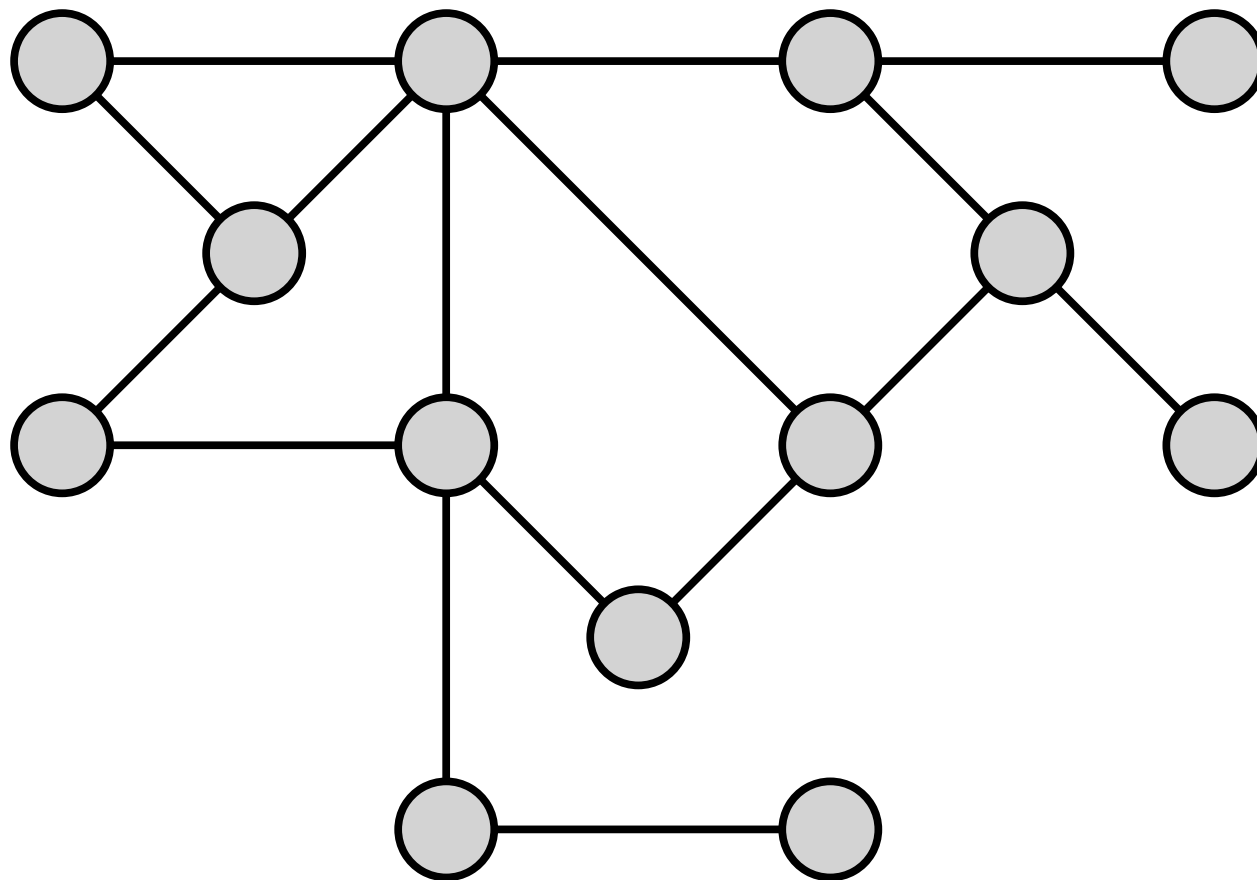
Choose a set of nodes, no two of which are adjacent.

Graphs

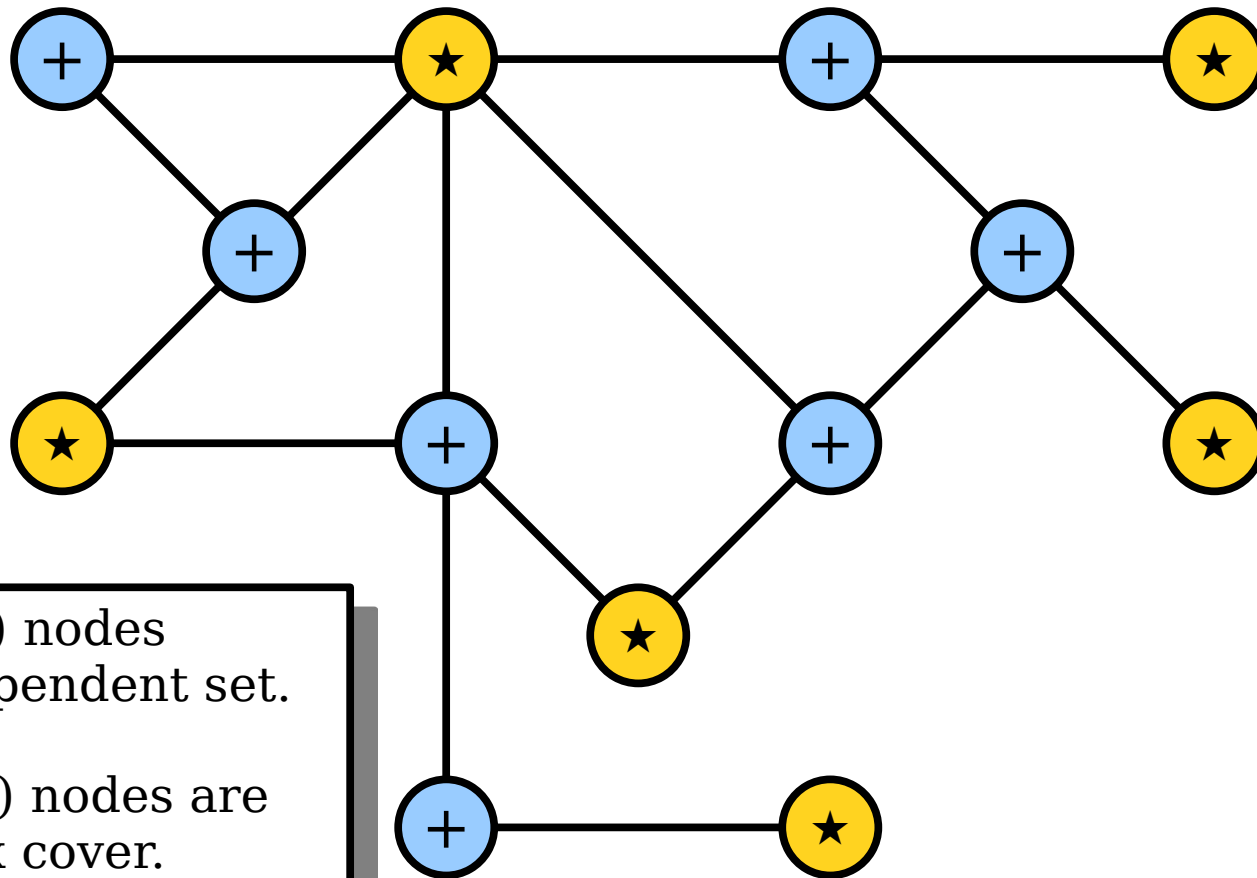
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A Connection

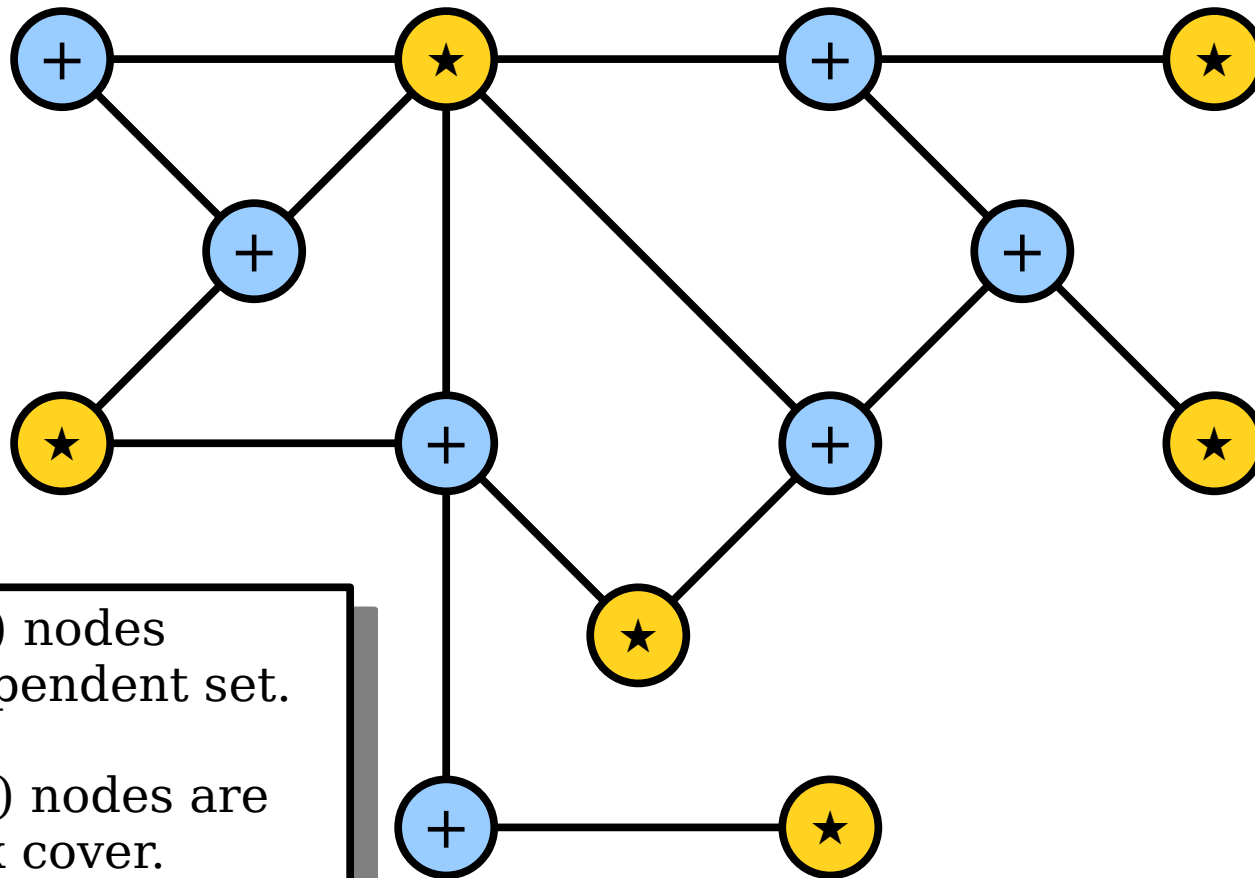


Independent sets and vertex covers are related.



- The star (★) nodes are an independent set.
- The plus (+) nodes are a vertex cover.

Independent sets and vertex covers are related.



■ The star (★) nodes are an independent set.

■ The plus (+) nodes are a vertex cover.

Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G if and only if $V - C$ is an independent set in G .

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G , then $V - C$ is an independent set in G .

What We're Assuming

G is a graph.

C is a vertex cover of G .

$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow u \in C \vee v \in C)$$

We're assuming a universally-quantified statement. That means we *don't do anything right now* and instead wait for an edge to present itself.

What We Need To Show

$V - C$ is an independent set in G .

$$\forall x \in V - C.$$
$$\forall y \in V - C.$$
$$\{x, y\} \notin E.$$

We need to prove a universally-quantified statement. We'll ask the reader to pick arbitrary choices of x and y for us to work with.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G , then $V - C$ is an independent set in G .

What We're Assuming

G is a graph.

C is a vertex cover of G .

$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow u \in C \vee v \in C)$$

$x \in V$ and $x \notin C$.

$y \in V$ and $y \notin C$.

What We Need To Show

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$$\forall x \in V - C.$$

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$x \in V$ and $x \notin C$.

$y \in V$ and $y \notin C$.

What We Need To Show

$V - C$ is an independent set in G .

$$\forall x \in V - C.$$

$$\forall y \in V - C.$$

$$\{x, y\} \notin E.$$



If this edge exists,
at least one of x
and y is in C .

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G , then $V - C$ is an independent set of G .

Proof: Assume C is a vertex cover of G . We need to show that $V - C$ is an independent set of G . To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We've reached a contradiction, so our assumption was wrong. Therefore, we have $\{x, y\} \notin E$, as required. ■

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set of G .

See appendix for this proof!

Be sure to check this one out and follow through with the negations of the statements above.

Finding an IS or VC

- The previous theorem means that finding a large IS in a graph is equivalent to finding a small VC.
 - If you've found one, you've found the other!
- **Open Problem:** Design an algorithm that, given an n -node graph, finds either the largest IS or smallest VC “efficiently,” where “efficiently” means “in time $O(n^k)$ for some $k \in \mathbb{N}$.”
 - There's a \$1,000,000 bounty on this problem – we'll see why in Week 10.

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Recap for Today

- A **graph** is a structure for representing items that may be linked together. **Digraphs** represent that same idea, but with a directionality on the links.
- Graphs can't have **self-loops**; digraphs can.
- **Vertex covers** and **independent sets** are useful tools for modeling problems with graphs.
- The complement of a vertex cover is an independent set, and vice-versa.

Next Time

- ***Paths and Trails***
 - Walking from one point to another.
- ***Local Area Networks***
 - The building blocks of the internet.
- ***Trees***
 - A fundamental class of graphs.

Appendix:

*Vertex Cover and Independent
Sets Lemma!*

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set of G .

Taking Negations

- What is the negation of this statement, which says “ C is a vertex cover?”

$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow \\ u \in C \vee v \in C \\)$$

Taking Negations

- What is the negation of this statement, which says “ C is a vertex cover?”

$$\begin{aligned} &\exists u \in V. \exists v \in V. (\{u, v\} \in E \wedge \\ &\quad u \notin C \quad \wedge \quad v \notin C \\ &)\end{aligned}$$

- This says “there is an edge where both endpoints aren’t in C .”

Taking Negations

- What is the negation of this statement, which says “ $V - C$ is an independent set?”

$$\forall u \in V - C. \forall v \in V - C. \{u, v\} \notin E$$

Taking Negations

- What is the negation of this statement, which says “ $V - C$ is an independent set?”

$$\exists u \in V - C. \exists v \in V - C. \{u, v\} \in E$$

- This says “there are two adjacent nodes in $V - C$.”

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set in G .

What We're Assuming

G is a graph.

C is not a vertex cover of G .

$$\exists u \in V. \exists v \in V. (\{u, v\} \in E \wedge u \notin C \wedge v \notin C)$$

We're assuming an existentially-quantified statement, so we'll *immediately* introduce variables u and v .

What We Need To Show

$V - C$ is not an ind. set in G .

$$\exists x \in V - C.$$

$$\exists y \in V - C.$$

$$\{x, y\} \in E.$$

We're proving an existentially-quantified statement, so we *don't* introduce variables x and y . We're on a scavenger hunt!

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set in G .

What We're Assuming

G is a graph.

C is not a vertex cover of G .

$$u \in V - C.$$

$$v \in V - C.$$

$$\{u, v\} \in E.$$

What We Need To Show

$V - C$ is not an ind. set in G .

$$\exists x \in V - C.$$

$$\exists y \in V - C.$$

$$\{x, y\} \in E.$$

Any ideas about
what we should
pick x and y to
be?

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set of G .

Proof: Assume C is not a vertex cover of G . We need to show that $V - C$ is not an independent set of G .

Since C is not a vertex cover of G , we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.

This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of G , as required. ■